Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: applications and interpretation
Higher level
Paper 3

Tuesday 8 November 2022 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

8 pages

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Answer **both** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 29]

In this question, you will explore possible approaches to using historical sports results for making predictions about future sports matches.

Two friends, Peter and Helen, are discussing ways of predicting the outcomes of international football matches involving Argentina.

Peter suggests analysing historical data to help make predictions. He lists the results of the most recent 240 matches in which Argentina played, in chronological order, then considers blocks of four matches at a time. He counts how many times Argentina has won in each block. The following table shows his results for the 60 blocks of four matches.

Number of wins for Argentina (per block of four matches)	Frequency
0	0
1	11
2	21
3	21
4	7

(a) Determine the mean number of wins per block of four matches for Argentina. [2]

Peter thinks that this data can be modelled by a binomial distribution with $n = 4$ and decides to carry out a χ^2 goodness of fit test.

(b) Use Peter's data to write down an estimate for the probability p for this binomial model. [1]

(c) (i) Use the binomial model to find the probability that Argentina win zero matches in a block of four matches. [1]

(ii) Find the expected frequency for zero wins. [2]

(This question continues on the following page)

Turn over

(Question 1 continued)

As some expected frequencies are less than 5, Peter combines rows in his table to produce the following observed frequencies. He then uses his binomial model to find appropriate expected frequencies, correct to one decimal place.

Number of wins for Argentina (per block of four matches)	Observed frequency	Expected frequency
0 or 1	11	10.8
2	21	20.7
3	21	20.7
4	7	7.8

(d) Peter uses this table to carry out a χ^2 goodness of fit test, to test the hypothesis that the data follows a binomial distribution with $n = 4$, at the 5% significance level.

For this test, state

(i) the null hypothesis; [1]

(ii) the number of degrees of freedom; [1]

(iii) the p-value; [2]

(iv) the conclusion, justifying your answer. [2]

(e) Using Peter's binomial model, find the probability that Argentina will win at least one of their next four international football matches. [2]

(This question continues on the following page)

(Question 1 continued)

Helen thinks that a better prediction might be made by considering the transition between matches. To keep the model simple, she decides to use only two states: Argentina won (A) or Argentina did not win (B). Helen looks at Peter's list of results and counts the number of times that:

- Argentina won, twice in succession (AA),
- Argentina won, then did not win (AB),
- Argentina did not win, then won (BA),
- Argentina did not win, twice in succession (BB).

She recorded the following results.

Transition	**Frequency**
AA	85
AB	60
BA	62
BB	32

Helen uses the relative frequencies to estimate the probabilities in a transition matrix.

(f) (i) Given that Argentina won the previous match, show that Helen's estimate for the probability of Argentina winning the next match is $\frac{17}{29}$. [2]

(ii) Write down the transition matrix, $\boldsymbol{T}$, for Helen's model. [2]

(g) (i) Show that the characteristic polynomial of $\boldsymbol{T}$ is $1363\lambda^2 - 1263\lambda - 100 = 0$. [3]

(ii) Hence or otherwise, find the eigenvalues of $\boldsymbol{T}$. [1]

(iii) Find the corresponding eigenvectors. [3]

(h) In her retirement, many years from now, Helen is planning to travel to three consecutive international football matches involving Argentina. Use Helen's model to find the probability that Argentina will win all three matches. [4]

2. [Maximum mark: 26]

Some medical conditions require patients to take medication regularly for long periods of time. In this question, you will explore the concentration of a medicinal drug in the body, when the drug is given repeatedly.

Once a drug enters the body, it is absorbed into the blood. As the body breaks down the drug over time, the concentration of the drug decreases. Let $C(t)$, measured in milligrams per millilitre (mg ml^{-1}), be the concentration of the drug, t hours after the drug is given to the patient. The rate at which the drug is broken down is modelled as directly proportional to its concentration, leading to the differential equation

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -kC\text{, where } k \in \mathbb{R}^{+}.$$

The initial concentration is $d\ \text{mg ml}^{-1}$, $d > 0$.

(a) By solving the differential equation, show that $C = d\mathrm{e}^{-kt}$. [3]

For the remainder of this question, you will consider a particular drug where it is known that $k = 0.2$. The first dose is given at time $t = 0$ and it is assumed that before this there is no drug present in the blood.

(b) Find the time, in hours, for this drug to reach 5% of its initial concentration. [2]

The drug is to be given every T hours and in constant doses, such that the concentration of the drug is increased by an amount $d\ \text{mg ml}^{-1}$. To simplify the model, it is assumed that each time the drug is given the concentration of the drug in the blood increases instantaneously.

(c) Show that the concentration of the drug is $d\left(1 + \mathrm{e}^{-0.2T} + \mathrm{e}^{-0.4T}\right)$ immediately after the third dose is given. [4]

Immediately after the nth dose is given, the concentration of the drug is

$$d\left(1 + \mathrm{e}^{-0.2T} + \mathrm{e}^{-0.4T} + \ldots + \mathrm{e}^{-0.2(n-1)T}\right).$$

(d) Show that this concentration can be expressed as $d\left(\dfrac{1 - \mathrm{e}^{-0.2nT}}{1 - \mathrm{e}^{-0.2T}}\right)$. [2]

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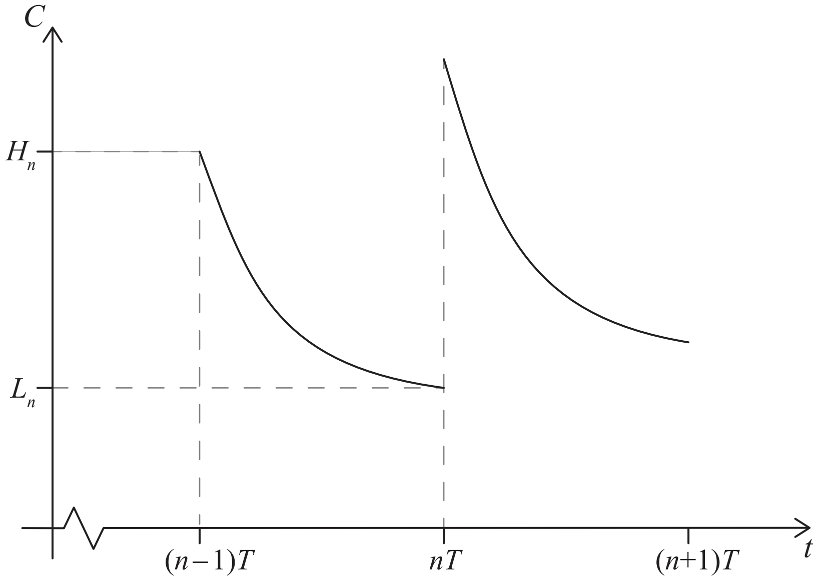
(Question 2 continued)

After a patient has been taking this drug for a long time, it is required to keep the concentration within a particular range so that it is both safe and effective.

Let H_n be the highest concentration of the drug in the body for the interval $(n-1)T \le t < nT$.

Let L_n be the lowest concentration of the drug in the body for the interval $(n-1)T \le t < nT$. This is shown in the following graph.



H_∞ is defined as $\lim\limits_{n\to\infty} H_n$ and L_∞ is defined as $\lim\limits_{n\to\infty} L_n$.

(e) Find, in terms of d and T, an expression for

(i) H_∞. [2]

(ii) L_∞. [3]

(f) Show that

(i) $H_\infty - L_\infty = d$. [2]

(ii) $5\ln\left(\dfrac{H_\infty}{L_\infty}\right) = T$. [3]

(This question continues on the following page)

(Question 2 continued)

It is known that this drug is ineffective if the long-term concentration is less than $0.06\,\text{mg ml}^{-1}$ and safe if it never exceeds $0.28\,\text{mg ml}^{-1}$.

(g) Hence, for this drug, find a suitable value for

(i) d. [1]

(ii) T. [1]

(h) For the values of d and T found in part (g), find the proportion of time for which the concentration of the drug is at least $0.06\,\text{mg ml}^{-1}$ between the first and second doses. [2]

(i) Suggest a reason why the instructions on the label of the drug might use a different value for T to that found in part (g)(ii). [1]

References: